The final speed is specified to be the stall speed at zero bank angle, and hence $u_f = 0.3575$. The isochrones are plotted in Fig. 1b. In some trajectories, the optimal bank angle reaches its upper bound of 78.5 deg, which is enforced by the maximum load factor constraint $n_{max} = 5$. It is seen that near the upper right corner of Fig. 1b , where both ψ_0 and x_t are large, the maximum endurance is penalized rather substantially.

Conclusion

This Note solves the most general form of the problem of maximum endurance gliding flight in a horizontal plane with the isochrones presented. The penalty on the maximum endurance is made evident when both the initial velocity yaw angle and the target distance are large.

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Stability Analysis of Gyroscopic Systems by Matrix Methods

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I. Introduction

THE application of matrix methods to a stability analysis of multiple-degree-of-freedom gyroscopic systems often provides insight into the interaction of the design parameters which otherwise may be overlooked. This additional behavioral information may in turn lead to a more informed selection of critical parameter values.

As an example of a useful matrix stability requirement, a system is said to be stable if its stiffness matrix is positive definite. Therefore, by requiring some generic stiffnes matrix to be positive definite, a set of linear inequalities in the system parameters is provided which aids in the selection process. In the case of systems described by gyroscopic forces, there are other combinations of the coefficient matrices which lead to useful stability results.

The intent of this work is to illustrate some simple stability conditions in matrix form, and, by way of example, to apply these requirements to the linear equations of motion of a dynamically tuned gyroscope (DTG). Finally, the derived regions of DTG stability are compared with those reported in the literature.[1]

In the next section, a matrix description of gyroscopic systems is given.

II. Matrix Description of Gyroscopic Systems

The gyroscopic systems of interest here are modeled by the matrix equation

$$M\ddot{x} + G\dot{x} + Kx = 0 \tag{1}$$

where $x = x(t)$ is an $n\times 1$ displacement vector, $\dot{x}$ an $n\times 1$ vector of velocities, $\ddot{x}$ an $n\times 1$ acceleration vector, M an $n\times n$ symmetric and positive definite matrix of mass and inertia parameters, K an $n\times n$ symmetric stiffness matrix, and G an $n\times n$ skew-symmetric matrix of gyroscopic force terms.

Equation (1) is used to describe the linear vibrations of many undamped gyroscopic systems in rotating reference frames. When $n=2$ this relationship depicts the equation of motion of a simple gyroscope.

The problem addressed here is to determine conditions on the elements of matrices M, G, and K which, when satisfied, insures a stable system performance. Matrix methods for stability analysis are discussed in the following section.

III. Stability by Matrix Methods

There are several well-known stability results available for Eq. (1). For example, if K is positive definite then the equilibrium position of Eq. (1) is stable. However, the system is not necessarily unstable if K is negative definite or indefinite (cf., Ref. 2).

To examine some of these possibilities it is convenient first to initially transform Eq. (1) into the equivalent form given by

$$I\ddot{y} + \tilde{G}\dot{y} + \tilde{K}y = 0 \tag{2}$$

where I is the $n\times n$ identity matrix, $y = M^{1/2}x$,

$$\tilde{G} = M^{-1/2}GM^{-1/2} = -\tilde{G}^T \quad \text{and} \quad \tilde{K} = M^{-1/2}KM^{-1/2} = \tilde{K}^T$$

Here the superscript T denotes the transpose of a matrix and the superscript $-1/2$ indicates the inverse of the positive definite square root of a positive definite matrix. Hagedorn[3] has shown that if the matrix $4\tilde{K} - \tilde{G}^2$ is negative definite, Eq. (2) and, hence, Eq. (1) is unstable.

A new stability result is available for a two-degree-of-freedom ($n=2$) system. Namely, if the determinant of $\tilde{K}$ is positive and if the trace of $4\tilde{K} - \tilde{G}^2$ is positive, then Eq. (2) and, hence, Eq. (1) is stable. This can also be stated in the original coordinates. Namely, if K is negative definite with a positive determinant and if the trace of M^{-1} $(4K - GM^{-1}G)$ is positive then Eq. (1) is stable.

To verify this conjecture for the $n=2$ case, consider the eigenvalue problem associated with Eq. (2). The characteristic equation is given by

$$\det(\lambda^2 I + \lambda\tilde{G} + \tilde{K}) = 0 \tag{3}$$

where λ is the eigenvalue and $\det(\cdot)$ denotes the determinant. Since $\tilde{K}$ is a real symmetric matrix, there exists a nonsingular orthogonal matrix S such that $S^TS = I$ and $S^T\tilde{K}S = \Lambda$, a diagonal matrix. Pre- and postmultiplying Eq. (3) by S^T and S, respectively, then yields

$$\det(\lambda^2 I + \lambda S^T\tilde{G}S + \lambda) = 0 \tag{4}$$

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The matrix $S^T\tilde{G}S$ is still skew symmetric since $(S^T\tilde{G}S)^T = -S^T\tilde{G}S$. therefore, if we let $g>0$ represent the nonzero elements of $S^T\tilde{G}S$ and ℓ_1 and ℓ_2 the elements of Λ, Eq. (4) produces

$$\lambda^2 = -½(\ell_1+\ell_2-g^2) \pm ½[(\ell_1+\ell_2+g^2)^2-4\ell_1\ell_2]^{½} \tag{5}$$

for the eigenvalues of Eq. (2).

If each λ^2 is a negative real number, each natural frequency will be a pure imaginary number and the gyroscopic system described by Eq. (2) will be stable about its equilibrium. To this end, consider the underlying assumptions made.

Since $\det\tilde{K}$ is positive and $\tilde{K}$ is not positive definite, the elements (eigenvalues of K) ℓ_1 and ℓ_2 both must be negative (i.e., K is actually negative definite) so that

$$\det K = \det\tilde{K} = \det\Lambda = \ell_1\ell_2 > 0 \tag{6}$$

and

$$\text{trace}K = \text{trace}\Lambda = \ell_1+\ell_2 < 0 \tag{7}$$

In addition, since the trace of a matrix is invariant under similarity transformations,[4] the trace condition yields

$$\text{trace}(4\tilde{K}-\tilde{G}^2) = \text{trace}(4S^T\tilde{K}S - S^T\tilde{G}SS^T\tilde{G}S) \tag{8}$$

$$= \text{trace}\begin{bmatrix} 4\ell_1+g^2 & 0 \\ 0 & 4\ell_2+g^2 \end{bmatrix} \tag{9}$$

and, therefore,

$$2(\ell_1+\ell_2)+g^2>0 \tag{10}$$

Equations (6), (7), and (10) now can be used to show that λ^2 is negative and real. First Eq. (10) can be written as

$$\ell_1+\ell_2+g^2 > -(\ell_1+\ell_2) > 0 \tag{11}$$

upon using Eq. (7). This last inequality taken together with Eq. (6) and compared to Eq. (5) shows that λ^2 is negative as long as the discriminant in Eq. (5) is positive. To see that the discriminant is positive, multiply Eq. (10) by $g^2>0$ and add the positive number $(\ell_1-\ell_2)^2$ to the left-hand side. This yields

$$(\ell_1-\ell_2)^2+2(\ell_1+\ell_2)g^2+g^4>0$$

which becomes

$$(\ell_1+\ell_2)^2-4\ell_1\ell_2+2(\ell_1+\ell_2)g^2+g^4>0$$

or

$$(\ell_1+\ell_2+g^2)(\ell_1+\ell_2+g^2)>4\ell_1\ell_2 \tag{12}$$

Comparison of inequality equation (12) with Eq. (5) indicates that λ^2 is real; since λ^2 is negative and real the system is stable. Note that the conditions given by the trace requirement [Eq. (10)] in general should be easier to apply than those derived from performing an actual eigenvalue analysis [as in inequality equation (7)].

IV. Application to the Dynamically Tuned Gyrscope

The dynamically tuned gyroscope represents a prime example of a two-degree-of-freedom system whose performance is influenced by overall geometry and the appropriate selection of inertias, stiffnesses, and rotor speed.[5] Application of matrix methods to DTG stability analysis is beneficial due to the possibility of certain combinations of DTG system parameters yielding an unstable response.

The equation of motion for a DTG with two perpendicular gimbals is given by Eq. (1), and the element of the matrices M, G, and K are taken from Ref. 1 (with $\alpha_1=0$ deg and $\alpha_2=90$ deg in the notation of the reference) and reproduced below.

$$M=\begin{bmatrix} A+a_1 & 0 \\ 0 & B+a_2 \end{bmatrix}$$

$$G=N(A+B-C)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$K=\begin{bmatrix} k_{11}+k_{22} + N^2(C-B+c_1-b_1) & 0 \\ 0 & k_{12}+k_{21} + N^2(C-A+c_2-b_2) \end{bmatrix}$$

Here A, B, and C are the principal moments of inertia of the rotor, a_i, b_i, c_i the principal moments of inertia of the ith gimbal; k_{11} and k_{12} the torsional stiffness elements connecting the drive shaft in the first and second gimbal, respectively; k_{21} and k_{22} the torsional stiffness elements connecting the rotor to the first and second gimbal, respectively, and N the rotor speed. The coordinates in Eq. (1) are the rotor displacements. A simplified diagram illustrating a two-gimbal DTG and a derivation of Eq. (1) may be found in Ref. 1. Also note that since M is diagonal it is a simple matter to compute the matrices $\tilde{G}$ and $\tilde{K}$ shown in Eq. (2). These relationships will be useful when the new stability result is exercised. In Sec. V a matrix stability analysis of the DTG is performed and the derived regions of stability are compared with those obtained by other methods.

V. Stability Analysis

Burdess and Fox[1] indicate that the stable operation of the DTG is greatly influenced by the rotor shape and the judicious choice of gyro parameters. Their analysis examines the system characteristic equation in order to identify stability boundaries. In the following paragraphs the stability properties of the DTG are analyzed by the matrix conditions discussed in Sec. III. In doing so, we point out ranges of values leading to stable gyro operation which were not mentioned previously. The analysis in Ref. 1 identifies three cases (depending on rotor shape). and discusses the range of rotor speeds that lead to stable DTG operation for each one. For the first situation, the rotor is a flat disk which is the configuration adopted in most practical designs. This corresponds to the case in which each of the terms $(C-A+c_2-b_2)$ and $(C-B+c_1-b_1)$ are both positive numbers, and the system is stable for all choices of rotor speed N.

The matrix approach draws the same conclusion since, if $(C-A+c_2-b_2)$ and $(C-B+c_1-b_1)$ are both positive, then the matrix K in Eq. (1) is positive definite for all N. However, the matrix condition (K positive definite) leads to a broader choice of gimbal and rotor inertias for stable operation. That is, if $(C-A+c_2-b_2)$ and $(C-B+c_1-b_1)$ are negative, the K matrix may still be positive definite for a certain range of rotor speeds N. Requiring K to be positive definite yields

$$k_{11}+k_{22}+N^2(C-B+c_1-b_1)>0 \tag{13}$$

$$k_{12}+k_{21}+N^2(C-A+c_2-b_2)>0 \tag{14}$$

Hence, for values of N such that

$$N>\min\left[\left(\frac{-(k_{11}+k_{22})}{(C-B+c_1-b_1)}\right)^{½},\ \left(\frac{-(k_{12}-k_{21})}{(C-A+c_2-b_2)}\right)^{½}\right] \tag{15}$$

the system is stable provided $(C-A+c_2-b_2)$ and $(C-B+c_1-b_1)$ are negative. Furthermore, K will be positive definite if

$$C-A+c_2-b_2>0, \qquad C-B+c_1-b_1>0$$

$$N<\left(\frac{-(k_{11}+k_{22})}{(C-B+c_1-b_1)}\right)^{1/2} \tag{16}$$

or, if

$$C-A+c_2-b_2<0, \qquad C-B+c_1-b_1<0$$

$$N<\left(\frac{-(k_{12}+k_{21})}{(C-A+c_2-b_2)}\right)^{1/2} \tag{17}$$

Either set of inequalities yields a stable operation and both follow directly from the positive definiteness of the matrix K in Eq. (1). They also represent the second and third situations discussed in Ref. 1, where it is required to calculate the roots of the stability boundary equation to arrive at the same conclusions. It should be noted that these situations occur when the rotor does not spin about a principal axis of minimum or maximum inertia.

In addition to the three cases analyzed in Ref. 1, there is yet another range of permissible rotor speeds and parameter values which insures stable DTG operation. This situation is identified by exercising the new matrix condition given in Sec. III, namely, $\det(\tilde{K})>0$ and trace $(4\tilde{K}-\tilde{G}^2)>0$, with $\tilde{K}$ negative definite. This requires that Eq. (1) be transformed into the system described by Eq. (2). Requiring $\det(\tilde{K})>0$ and $\tilde{K}$ to be negative definite yields

$$k_{11}+k_{22}+2N^2(C-B+c_1-b_1)<0 \tag{18}$$

$$k_{12}+k_{21}+2N^2(C-A+c_2-b_2)<0 \tag{19}$$

while the trace of $4\tilde{K}-\tilde{G}^2$ is given by

$$4\frac{k_{11}+k_{22}+N^2(C-B+c_1-b_1)}{A+a_1}+2\frac{N^2(A+B-C)^2}{(A+a_1)(B+a_2)}$$
$$+4\frac{k_{12}+k_{21}+N^2(C-A+c_2-b_2)}{B+a_2} \tag{20}$$

Demanding that Eq. (20) be positive yields

$$4(k_{11}+k_{22})(B+a_2)+4(k_{12}+k_{21})(A+a_1)$$
$$+2N^2[(A+B-C)^2$$
$$+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)]>0 \tag{21}$$

which results in two subcases. The first case is

$$(A+B-C)^2+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)>0 \tag{22}$$

and the DTG is stable for all values of the rotor speed N. The other subcase occurs when the inequality sign in Eq. (22) is reversed. For this condition, the DTG is stable, provided that N is such that

$$N<[-2[(k_{11}+k_{22})(B+a_2)+(k_{12}+k_{21})(A+a_1)]^{1/2}$$
$$\div[(A+B-C)^2+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)] \tag{23}$$

It should be noted that the existence of additional stable regions for the case when $\tilde{K}$[from Eq. (2)] is not negative definite is a result of the stabilizing effect of gyroscopic forces.

VI. Conclusion

A simple matrix stability condition is derived for linear gyroscopic systems with two degrees of freedom. The matrix method of analysis requires less calculation than the usual procedures of generating the system response or solving the associated eigenvalue problem. Also, matrix techniques have the ability to indicate subtleties in the design choices which otherwise may be undiscovered. The new matrix condition along with standard matrix stability requirements are applied to a model of a dynamically tuned gyroscope. It is shown that the matrix approach yields a larger range of parameter values to choose from for the stable dynamically tuned gyroscope operation than previously indicated in the literature. It is hoped that the new matrix condition may find application in the general class of complex gyroscopic systems which includes the DTG.

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Model Reduction of Control Systems

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Introduction

THE analysis and design of large-order control systems is quite tedious and costly. Therefore, it is desirable to replace a given large-order system with a lower-order system in such a way that the lower-order system retains the significant characteristics of the given system.

From the standpoint of flying qualities, the usefulness of the model reduction technique lies in verifying compliance with the specifications. The analytical description of a typical augmented aircraft system is usually quite complicated. Thus, it becomes difficult to check whether or not the designed

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